Almost Chebyshev Subsets in Reflexive Banach Spaces

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§1. Let K be a nonempty subset in a (real) Banach space X. For each $x \in X$, we say that $y \in K$ is a *best approximation* from x to K if $\|x - y\| = \inf \{\|x - z\|: z \in K\}$. A set K is called *proximinal* (*Chebyshev*) if every point $x \in X$ admits a (unique) best approximation from K. It is easy to see that in a reflexive Banach space every weakly closed subset is proximinal; however, the statement is not true for arbitrary (norm) closed sets. In [8] Stečkin proved that *for any closed subset K in a uniformly convex space, the set of points in X which fail to have unique best approximation from K is a set of first category*. We call a set with the above property *almost Chebyshev*. This concept has been studied by many authors (*cf. e.g.* [1], [2], [4], [5], [6]). A remaining unsolved problem is: Is every closed subset in a reflexive locally uniformly convex space almost Chebyshev [7, p. 375]? In this note, we give a positive answer. We remark that the answer is not true for reflexive strictly convex spaces [Edelstein, 2] or locally uniformly convex spaces [Cobzaş, 1].

§2. Let X be a Banach space. We use X^* to denote the dual of X. A real-valued function f on X is said to be *Fréchet differentiable* if for each $x \in X$, there exists an $x^* \in X^*$ such that for any $\epsilon > 0$, there exists $\eta > 0$ which satisfies

$$|f(y) - f(x) - \langle x^*, y - x\rangle| \leq \epsilon\|y - x\| \qquad \forall \|y - x\| < \eta.$$

For $x \in X$, a functional x^* in X^* is called a *local ϵ-support* of f at x if there exists an $\eta > 0$ such that

$$\|x - y\| < \eta \Rightarrow f(y) - f(x) \geq \langle x^*, y - x\rangle - \epsilon\|y - x\|.$$

Note that the x^* in the above definition depends on ϵ. In [3], Ekeland and Lebourg proved

Proposition 1. *Suppose X is a Banach space which admits a nonnegative Fréchet differentiable function with bounded nonempty support. Then for any $\epsilon > 0$ and for any lower semicontinuous function f on X, there exists a dense set of points in X where f is locally ϵ-supported.*

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Let K be a closed subset in a Banach space X. We define the distance function r from $x \in X$ to K as

$$r(x) = \inf\{\|x - z\| : z \in K\}, \qquad x \in K.$$

Note that $|r(y) - r(x)| \leq \|y - x\|$ for all $x, y \in X$. Furthermore, for $x \notin K$, $0 < \epsilon < \min\{r(x), 1\}$, we can find a $z \in K$ such that $\|x - z\| < r(x) + \epsilon^2$. Let y be a point on the line segment jointing x and z with $\|x - y\| = \epsilon$. We have

$$\frac{r(x) - r(y)}{\|x - y\|} \geq \frac{\|x - z\| - \|y - z\| - \epsilon^2}{\|x - y\|} = 1 - \epsilon.$$

Hence,

$$\overline{\lim_{y \to x}} \frac{r(x) - r(y)}{\|x - y\|} = 1, \qquad x \notin K.$$

Lemma 2. *Let K be a closed subset in a Banach space X. Suppose $x \notin K$ and suppose r is locally ϵ-supported by an x^* at x. Then* $|\,\|x^*\| - 1| \leq \epsilon$.

Proof. Let $\eta > 0$ be chosen as in the definition of local ϵ-support. We have for any $\|y - x\| < \eta$

$$\frac{r(x) - r(y)}{\|x - y\|} - \epsilon \leq \left\langle x^*, \frac{x - y}{\|x - y\|} \right\rangle \leq \|x^*\|$$

and

$$\left\langle x^*, \frac{y - x}{\|y - x\|} \right\rangle \leq \frac{r(y) - r(x)}{\|y - x\|} + \epsilon \leq 1 + \epsilon.$$

Since $\overline{\lim}_{y \to x} \dfrac{r(x) - r(y)}{\|x - y\|} = 1$, we have $1 - \epsilon \leq \|x^*\| \leq 1 + \epsilon$.

Let $B(x, d)$ denote the closed ball with center at x and radius d. For any closed subset K in X, $x \notin K$, $x^* \in X^*$, $\epsilon, \delta > 0$, we let

$$S(x, x^*, \epsilon, \delta) = \{z : z \in B(x, r(x) + \delta), \langle x^*, z - x\rangle \leq -r(x)(1 - \epsilon)\}$$

and

$$A_\epsilon = \{x \in X \setminus K : B(x, r(x) + \delta) \cap K \subseteq S(x, x^*, \epsilon, \delta) \quad \text{for some} \quad \delta > 0, |\,\|x^*\| - 1| < \epsilon\}.$$

Lemma 3. *Let X be a Banach space which admits a non-negative Fréchet differentiable function with non-empty bounded support. For $0 < \epsilon < \frac{1}{2}$, let A_ϵ be defined as above. Then A_ϵ is an open dense subset in $X \setminus K$.*

Proof. We first prove that A_ϵ is an open subset in $X \setminus K$. For $x \in A_\epsilon$, let x^*, δ be chosen as in the definition. We may assume further that the distance from

$K \cap B(x, r(x) + \delta)$ to $B(x, r(x) + \delta)\setminus S(x, x^*, \epsilon, \delta)$ is positive, say $\beta > 0$ (for otherwise, we can take $cx^*$with $c > 1$, $| \|cx^*\| - 1| < \epsilon$, in the definition of A_ϵ). Let $\alpha = \min\left\{\frac{\delta}{5}, \frac{\beta}{2}\right\}$; for $\|y - x\| < \alpha$, we let $y^* = x^*$. We need to show

$$(*) \qquad B(y, r(y) + \alpha) \cap K \subseteq S(y, y^*, \epsilon, \alpha) \qquad \forall \|y - x\| < \alpha.$$

Note that for $z \in B(y, r(y) + \alpha) \cap K$, $\|z - x\| < r(x) + 3\alpha$, hence $z + w \in B(x, r(x) + \delta)$ for all $\|w\| \leq 2\alpha$. That $\alpha \leq \frac{\beta}{2}$ implies $z + w \in S(x, x^*, \epsilon, \delta)$ for $\|w\| \leq 2\alpha$. It follows that

$$\langle x^*, z - x\rangle \leq -r(x)(1 - \epsilon) - 2\alpha\|x^*\|.$$

Now for $z \in B(y, r(y) + \alpha) \cap K$

$$\begin{aligned} \langle y^*, z - y\rangle &= \langle x^*, z - y\rangle \\ &\leq \langle x^*, z - x\rangle + \alpha\|x^*\| \\ &\leq -r(x)(1 - \epsilon) - 2\alpha\|x^*\| + \alpha\|x^*\| \\ &\leq -r(y)(1 - \epsilon) + \alpha(1 - \epsilon) - \alpha\|x^*\| \\ &\leq -r(y)(1 - \epsilon). \end{aligned}$$

Hence (*) is proved. To prove that A_ϵ is dense in $X\setminus K$, by Proposition 1, it suffices to show that if r is locally $\frac{\epsilon}{4}$-supported by x^* at x, then $x \in A_\epsilon$. Without loss of generality, we assume that $r(x) \leq 1$. Let $0 < \eta < 1$ be a number satisfying the definition of local $\frac{\epsilon}{4}$-support of r at x and let $\delta = \frac{\eta\epsilon \cdot r(x)}{4}$. For $z \in B(x, r(x) + \delta) \cap K$, we have $\left\|\frac{\eta}{2}(z - x)\right\| < \eta$, hence

$$\begin{aligned} &-\frac{\epsilon}{4}\cdot\frac{\eta}{2}\|z - x\| + \left\langle x^*, \frac{\eta}{2}(z - x)\right\rangle \\ &\leq r\left(\frac{\eta}{2}z + \left(1 - \frac{\eta}{2}\right)x\right) - r(x) \\ &\leq \left(1 - \frac{\eta}{2}\right)\|z - x\| - (\|z - x\| - \delta) \\ &\leq -\frac{\eta}{2}\|z - x\| + \delta \\ &\leq -\frac{\eta}{2}\cdot r(x) + \delta. \end{aligned}$$

Dividing the inequality by $\frac{\eta}{2}$, we have

$$\langle x^*, z - x\rangle \leq -r(x) + \frac{2\delta}{\eta} + \frac{\epsilon}{4}(r(x) + \delta)$$

$$\leq -r(x) + \frac{\epsilon}{2} r(x) + \frac{\epsilon}{2} r(x)$$

$$= -r(x)(1 - \epsilon).$$

This implies $B(x, r(x) + \delta) \cap K \subseteq S(x, x^*, \epsilon, \delta)$ and hence, by Lemma 2, $x \in A_\epsilon$.

A Banach space is said to have *property* (K) if for any sequence $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $x_n \to x$ in norm. It is well known that every locally uniformly convex space has this property. A subset K in X is called *almost proximinal (almost Chebyshev)* if the set of $x \in X$ which admits best approximation (unique) from K is a second category subset in X.

Theorem 4. *Let X be a reflexive Banach space with property (K). Then every closed subset in X is almost proximinal.*

Proof. It is clear that every point in K has a best approximation, hence, we need only consider $X \setminus K$. Recall that if X is reflexive, then X admits an equivalent Fréchet differentiable norm [9]. Lemma 3 implies that the set $A = \cap_{n>2} A_{\frac{1}{n}}$ is a dense G_δ subset in $X \setminus K$. For $x \in A$ and for each $n > 2$, choose $z_n \in B\left(x, r(x) + \frac{1}{n}\right) \cap K$. By the reflexivity, $\{z_n\}$ has a weakly converging subsequence. Without loss of generality, we assume that $z_n \to z$ weakly. Note that

$$z_n \in B\left(x, r(x) + \frac{1}{m}\right) \cap K \subseteq S\left(x, x_m^*, \frac{1}{m}, \delta_m\right) \qquad \forall m \leq n.$$

Hence for each fixed m,

$$\langle x_m^*, z_n - x\rangle \leq -r(x)\left(1 - \frac{1}{m}\right) \qquad \forall m \leq n.$$

This implies

$$\langle x_m^*, z - x\rangle \leq -r(x)\left(1 - \frac{1}{m}\right) \qquad \forall m$$

and

$$\|x_m^*\| \cdot \|z - x\| \geq r(x)\left(1 - \frac{1}{m}\right) \qquad \forall m.$$

We thus have $\lim_{n\to\infty} \|z_n - x\| = r(x) = \|z - x\|$.

Since X has property (K), we conclude that $(z_n - x) \to (z - x)$ in norm, *i.e.*

$z_n \to z$ in norm. As K is closed, $z \in K$ and is a best approximation from K to x.

Theorem 5. *Let X be a reflexive locally uniformly convex space, then every closed subset K in X is almost Chebyshev.*

Proof. It is proved in [8] that for any closed subset K in a locally uniformly convex space, the set of x which has not more than one (may be none) best approximation from K is a dense G_δ. Together with the above theorem, we conclude that every closed subset in such a space is almost Chebyshev.

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References

1. S. Cobzas, *Antiproximinal sets in Banach spaces*, Math. Balkanica **4** (1974), 79–82.
2. M. Edelstein, *Weakly proximinal sets*, J. Approximation Theory **18** (1976), 1–8.
3. I. Ekeland & G. Lebourg, *Generic Fréchet-differentiability and perturbed optimation problems in Banach spaces*, Trans. Amer. Math. Soc. **224** (1976), 193–216.
4. A. Garkavi, *On Chebyshev and almost Chebyshev subspaces*, Izv. Akad. Nauk. SSSR. Ser. Mat. **28** (1964), 799–818. Translated in Amer. Math. Soc. Transl. **96** (1970), 153–175.
5. K. Lau, *On almost Chebyshev Subspaces*, J. Approximation Theory **21** (1977), 319–327.
6. E. Rozema, *Almost Chebyshev subspaces of $L^1(\mu, E)$*, Pacific J. Math. **53** (1974), 585–604.
7. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer Verlag, 1970.
8. S. Stečkin, *Approximation properties of sets in normed linear spaces*, Rev. Math. Pures. Appl. **8** (1963), 5–18 (Russian).
9. S. Trojanski, *On locally convex and differentiable norm in certain non-separable Banach spaces*, Studia Math. **37** (1971), 173–180.

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